JEE	Class –12^{th}	Topic – Adjoint and Inverse of a Matrix

1. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then adj(adjA) is equal to

(a) A
(b) 1
(c) 0
(d) None of these

Sol. (a) We know that, $\text{adj}(\text{adj}A) = |A|^{n-2}A$, if $|A| \neq 0$

Since,

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Here, n = 3 (3 order matrix)

$$\therefore |A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$$
$$= 3(-3+4) + 3(2-0) + 4(-2-0)$$
$$= 1 \neq 0$$

∴ A is non-singular.

$$\text{adj}(\text{adj}A) = |A|^{3-2} \cdot A = A$$

2. Let A be a 3×3 matrix such that $\text{adj}A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & -1 \end{bmatrix}$ and B = adj(adjA). |f|A |= λ and $\left|(B^{-1})^T\right| = \mu$, then the ordered pair, $(|\lambda|, \mu)$ is equal to

(JEE Main 2020)

(a) (3,81)
(b) $\left(9, \frac{1}{81}\right)$
(c) $\left(9, \frac{1}{9}\right)$
(d) $\left(3, \frac{1}{81}\right)$

Sol. (d) It is given that, $\text{adj}(A) = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & -1 \end{bmatrix}$

$\Rightarrow |adjA| = 2(0+4) + 1(1-2) + 1(2) = 9$

$\because |adjA| = |A|^{3-1}$

$\Rightarrow |A|^2 = |adjA| \Rightarrow |A| = \pm 3 \Rightarrow |\lambda| = 3$

$\because B = adj(adjA)$

$\Rightarrow |B| = |A|^{3-1)^2} = |A|^4 = 81$

$\because \left|(B^{-1})^T\right| = |B^{-1}| = \dfrac{1}{|B|} = \dfrac{1}{81} \Rightarrow \mu = \dfrac{1}{81}$

$\therefore (|\lambda|, \mu)$ is $\left(3, \dfrac{1}{81}\right)$

Hence, option (d) is correct.

3. If the matrices $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$, $B = adj\ A$ and $C = 3A$, then $\frac{|adjB|}{|C|}$ is equal to

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(a) 16
(b) 2
(c) 8
(d) 72

Sol. (c) Given matrices $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$ $B = adj(A)$ and $C = 3A$

Now, $|adj(B)| = |\ adj(adj(A)\ |$

$[\because B = adj(A)]$

and $|\ adj\big(adj(A)\big| = |A|^{(n-1)^2}$, where n is the order of square matrix A.

$\therefore\ |adj(adj(A))| = |A|^{B-1|^2} = |A|^4$

and $|C| = |3A| = 3^3|A| = 27|A|$

$\because |KA| = K^n|A|$, where K is a scalar and n is the order of square matrix A.

$\therefore \frac{|adjB|}{|C|} = \frac{|A|^4}{27|A|} = \frac{|A|^3}{27} = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{vmatrix}$

$\because |A| = 1(9+4) - 1(3-4) + 2(-1-3) = 13 + 1 - 8 = 6$

So, $\frac{|adjB|}{|C|} = \frac{|A|^3}{27}$

$$= \frac{6^3}{27} = 2^3 = 8$$

4. The inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ is

(a) $\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

(b) $\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$

(c) $\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(d) None of these

5. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \frac{\pi}{12}$, is equal to

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(a) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

Sol. (c) We have, $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\therefore\ |A| = \cos^2\theta + \sin^2\theta = 1$

and $adjA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Now, $C_{11} = \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1, C_{23} = -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3;$

$C_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8, C_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1;$

$C_{13} = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5, C_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2;$

$C_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1, C_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -1$

$C_{22} = \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$

$\therefore$ Matrix of cofactors, $C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix}$

$\therefore$ $adj(A) = C' = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

Hence, $(A)^{-1} = \frac{adjA}{|A|}$

$$= -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

[:- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then adj $A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$]

$\Rightarrow$ $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\left[\because A^{-1} = \frac{adjA}{|A|}\right]$

Note that, $A^{-50} = (A^{-1})^{50}$

Now, $A^{-2} = (A^{-1})(A^{-1})$

$$\Rightarrow A^{-2} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & \cos\theta\sin\theta + \sin\theta\cos\theta \\ -\cos\theta\sin\theta - \cos\theta\sin\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

Also, $A^{-3} = (A^{-2})(A^{-1})$

$$A^{-3} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta \end{bmatrix}$$

Similarly,

$$A^{-50} = \begin{bmatrix} \cos 50\theta & \sin 50\theta \\ -\sin 50\theta & \cos 50\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{25}{6}\pi & \sin\frac{25}{6}\pi \\ -\sin\frac{25}{6}\pi & \cos\frac{25}{6}\pi \end{bmatrix} \left[\text{when } \theta = \frac{\pi}{12}\right]$$

$$= \begin{bmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}$$

$$\left[\begin{matrix} \because \cos\left(\frac{25\pi}{6}\right) = \cos\left(4\pi + \frac{\pi}{6}\right) = \cos\frac{\pi}{6} \\ \text{and } \sin\left(\frac{25\pi}{6}\right) = \sin\left(4\pi + \frac{\pi}{6}\right) = \sin\frac{\pi}{6} \end{matrix}\right]$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

6. If $A = \begin{bmatrix} x & 2 \\ 4 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{-1}{12} \\ \frac{-1}{6} & \frac{4}{9} \end{bmatrix}$, then find the value of x ?

Concept:

$A \times A^{-1} = I$, where I is an identity matrix

$$|A| = \frac{1}{|A - 1|}$$

Calculation:

Given: $\mathbf{A} = \begin{bmatrix} x & 2 \\ 4 & 3 \end{bmatrix}$ and $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{-1}{12} \\ \frac{-1}{6} & \frac{4}{9} \end{bmatrix}$

$$|A^{-1}| = \frac{4}{72} - \frac{1}{72} = \frac{3}{72} = \frac{1}{24}$$

$$|A| = \frac{1}{|A^{-1}|} = 24$$

$\Rightarrow 3x - 8 = 24$

$\therefore x = \dfrac{32}{3}$

7. If $A^2 - 2A - 1 = 0$, then inverse of A is

(a) 1

(b) $A + 2$

(c) $A - 2$

(d) A

Calculation:

Given: $A^2 - 2A - 1 = 0$

$\Rightarrow AA - 2A = 1$

Post multiply by A^{-1}, we get

$\Rightarrow AAA^{-1} = 2AA^{-1} = IA^{-1}$
$\Rightarrow AI - 2I = A^{-1}$ [$\because AA^{-1} = A^{-1}A = I$]
$\therefore A^{-1} = A - 2$

the inverse of A is A - 2